# *Comment on "A Recursive Parametrisation of Unitary Matrices"*

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**Abstract**

In this note we make a comment on the paper by Jarlskog (math–ph/0504049) in which she gave a parametrisation to unitary matrices. Namely, we show that her recursive method is essentially obtained by the canonical coordinate of the second kind in the Lie group theory, and propose a problem on constructing unitary gates in quantum computation by making use of the parametrisation.

In the paper [1] Jarlskog gave an interesting parametrisation to unitary matrices. See also [2] as a similar parametrisation. In this note we show that her recursive method is essentially obtained by the so–called canonical coordinate of the second kind in the Lie group theory.

The unitary group is defined as

$$U(n) = \left\{ U \in M(n, \mathbf{C}) \mid U^{\dagger}U = UU^{\dagger} = \mathbf{1}_n \right\} \tag{1}$$

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and its (unitary) algebra is given by

$$u(n) = \left\{ X \in M(n, \mathbf{C}) \mid X^\dagger = -X \right\}. \tag{2}$$

Then the exponential map is

$$\exp : u(n) \ \longrightarrow \ U(n) \ ; \ X \ \mapsto \ U \equiv \mathrm{e}^X. \tag{3}$$

This map is canonical but not easy to calculate.

We write down the element $X \in u(n)$ explicitly :

$$X = \begin{pmatrix} i\theta_1 & z_{12} & z_{13} & \cdots & z_{1,n-1} & z_{1n} \\ -\bar{z}_{12} & i\theta_2 & z_{23} & \cdots & z_{2,n-1} & z_{2n} \\ -\bar{z}_{13} & -\bar{z}_{23} & i\theta_3 & \cdots & z_{3,n-1} & z_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\bar{z}_{1,n-1} & -\bar{z}_{2,n-1} & -\bar{z}_{3,n-1} & \cdots & i\theta_{n-1} & z_{n-1,n} \\ -\bar{z}_{1,n} & -\bar{z}_{2,n} & -\bar{z}_{3,n} & \cdots & -\bar{z}_{n-1,n} & i\theta_n \end{pmatrix}. \tag{4}$$

This $X$ is decomposed into

$$X = X_0 + X_2 + \cdots + X_j + \cdots + X_n$$

where

$$X_0 = \begin{pmatrix} i\theta_1 & & & & & \\ & i\theta_2 & & & & \\ & & i\theta_3 & & & \\ & & & \ddots & & \\ & & & & i\theta_{n-1} & \\ & & & & & i\theta_n \end{pmatrix} \tag{5}$$

and for $2 \le j \le n$

$$X_j = \begin{pmatrix} 0 & & & & & \\ & \ddots & & |z_j\rangle & & \\ & & \ddots & & & \\ & -\langle z_j| & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad |z_j\rangle = \begin{pmatrix} z_{1j} \\ z_{2j} \\ \vdots \\ z_{j-1,j} \end{pmatrix}. \tag{6}$$

Then the canonical coordinate of the second kind in the unitary group (Lie group) is well–known and given by

$$u(n) \ni X = X_0 + X_2 + \cdots + X_j + \cdots + X_n \longrightarrow \mathrm{e}^{X_0}\mathrm{e}^{X_2}\cdots\mathrm{e}^{X_j}\cdots\mathrm{e}^{X_n} \in U(n) \tag{7}$$

in this case [1]. Therefore let us calculate $\mathrm{e}^{X_j}$ for $j \ge 2$ ($j = 0$ is trivial). From

$$X_j = \begin{pmatrix} \mathbf{0}_{j-1} & |z_j\rangle & \\ -\langle z_j| & 0 & \\ & & \mathbf{0}_{n-j} \end{pmatrix} \equiv \begin{pmatrix} K & \\ & \mathbf{0}_{n-j} \end{pmatrix} \quad \Longrightarrow \quad \mathrm{e}^{X_j} = \begin{pmatrix} \mathrm{e}^K & \\ & \mathbf{1}_{n-j} \end{pmatrix}$$

we have only to calculate the term $\mathrm{e}^K$, which is easy task. From

$$K = \begin{pmatrix} \mathbf{0}_{j-1} & |z_j\rangle \\ -\langle z_j| & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} -|z_j\rangle\langle z_j| & \\ & -\langle z_j|z_j\rangle \end{pmatrix},$$

$$K^3 = \begin{pmatrix} \mathbf{0}_{j-1} & -\langle z_j|z_j\rangle|z_j\rangle \\ \langle z_j|z_j\rangle\langle z_j| & 0 \end{pmatrix} = -\langle z_j|z_j\rangle K \tag{8}$$

we have important relations

$$K^{2n+1} = \left(-\langle z_j|z_j\rangle\right)^n K, \quad K^{2n+2} = \left(-\langle z_j|z_j\rangle\right)^n K^2 \quad \text{for} \quad n \ge 0.$$

Therefore

$$\mathrm{e}^K = \mathbf{1}_j + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} X^{2n+2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} X^{2n+1}$$

[1] There are of course some variations

$$
\begin{aligned}
&= \mathbf{1}_j + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} \left(-\langle z_j|z_j\rangle\right)^n K^2 + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(-\langle z_j|z_j\rangle\right)^n K \\
&= \mathbf{1}_j + \sum_{n=0}^{\infty} (-1)^n \frac{\left(\sqrt{\langle z_j|z_j\rangle}\right)^{2n}}{(2n+2)!} K^2 + \sum_{n=0}^{\infty} (-1)^n \frac{\left(\sqrt{\langle z_j|z_j\rangle}\right)^{2n}}{(2n+1)!} K \\
&= \mathbf{1}_j - \frac{1}{\langle z_j|z_j\rangle} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(\sqrt{\langle z_j|z_j\rangle}\right)^{2n+2}}{(2n+2)!} K^2 + \frac{1}{\sqrt{\langle z_j|z_j\rangle}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\sqrt{\langle z_j|z_j\rangle}\right)^{2n+1}}{(2n+1)!} K \\
&= \mathbf{1}_j - \frac{1}{\langle z_j|z_j\rangle} \sum_{n=1}^{\infty} (-1)^n \frac{\left(\sqrt{\langle z_j|z_j\rangle}\right)^{2n}}{(2n)!} K^2 + \frac{1}{\sqrt{\langle z_j|z_j\rangle}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\sqrt{\langle z_j|z_j\rangle}\right)^{2n+1}}{(2n+1)!} K \\
&= \mathbf{1}_j - \frac{1}{\langle z_j|z_j\rangle} \left(\cos(\sqrt{\langle z_j|z_j\rangle}) - 1\right) K^2 + \frac{\sin(\sqrt{\langle z_j|z_j\rangle})}{\sqrt{\langle z_j|z_j\rangle}} K \\
&= \mathbf{1}_j + \left(1 - \cos(\sqrt{\langle z_j|z_j\rangle})\right) \frac{1}{\langle z_j|z_j\rangle} K^2 + \sin(\sqrt{\langle z_j|z_j\rangle}) \frac{1}{\sqrt{\langle z_j|z_j\rangle}} K.
\end{aligned}
$$

If we define a normalized vector as

$$
|\tilde{z}_j\rangle = \frac{1}{\sqrt{\langle z_j|z_j\rangle}} |z_j\rangle \implies \langle \tilde{z}_j|\tilde{z}_j\rangle = 1
$$

then

$$
\frac{1}{\sqrt{\langle z_j|z_j\rangle}} K = \begin{pmatrix} \mathbf{0}_{j-1} & |\tilde{z}_j\rangle \\ -\langle \tilde{z}_j| & 0 \end{pmatrix}, \quad \frac{1}{\langle z_j|z_j\rangle} K^2 = \begin{pmatrix} -|\tilde{z}_j\rangle\langle \tilde{z}_j| & \\ & -1 \end{pmatrix}.
$$

Therefore

$$
\mathrm{e}^{K} = \begin{pmatrix} \mathbf{1}_{j-1} - \left(1 - \cos(\sqrt{\langle z_j|z_j\rangle})\right) |\tilde{z}_j\rangle\langle \tilde{z}_j| & \sin(\sqrt{\langle z_j|z_j\rangle})|\tilde{z}_j\rangle \\ -\sin(\sqrt{\langle z_j|z_j\rangle})\langle \tilde{z}_j| & \cos(\sqrt{\langle z_j|z_j\rangle}) \end{pmatrix}. \tag{9}
$$

As a result we obtain

$$
\mathrm{e}^{X_j} = \begin{pmatrix} \mathbf{1}_{j-1} - \left(1 - \cos(\sqrt{\langle z_j|z_j\rangle})\right) |\tilde{z}_j\rangle\langle \tilde{z}_j| & \sin(\sqrt{\langle z_j|z_j\rangle})|\tilde{z}_j\rangle & \\ -\sin(\sqrt{\langle z_j|z_j\rangle})\langle \tilde{z}_j| & \cos(\sqrt{\langle z_j|z_j\rangle}) & \\ & & \mathbf{1}_{n-j} \end{pmatrix}. \tag{10}
$$

This is just the matrix given in [1].

We make a comment on the case of $n = 2$. Since

$$
|\tilde{z}\rangle = \frac{z}{|z|} \equiv \mathrm{e}^{i\alpha}, \quad \langle \tilde{z}| = \frac{\bar{z}}{|z|} = \mathrm{e}^{-i\alpha} \implies |\tilde{z}\rangle\langle \tilde{z}| = \langle \tilde{z}|\tilde{z}\rangle = 1,
$$

we have

$$
\begin{aligned}
\mathrm{e}^{X_0}\mathrm{e}^{X_2} &= \begin{pmatrix} \mathrm{e}^{i\theta_1} & \\ & \mathrm{e}^{i\theta_2} \end{pmatrix} \begin{pmatrix} \cos(|z|) & \mathrm{e}^{i\alpha}\sin(|z|) \\ -\mathrm{e}^{-i\alpha}\sin(|z|) & \cos(|z|) \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{i\theta_1} & \\ & \mathrm{e}^{i\theta_2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\alpha/2} & \\ & \mathrm{e}^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos(|z|) & \sin(|z|) \\ -\sin(|z|) & \cos(|z|) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-i\alpha/2} & \\ & \mathrm{e}^{i\alpha/2} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{e}^{i(\theta_1+\alpha/2)} & \\ & \mathrm{e}^{i(\theta_2-\alpha/2)} \end{pmatrix} \begin{pmatrix} \cos(|z|) & \sin(|z|) \\ -\sin(|z|) & \cos(|z|) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-i\alpha/2} & \\ & \mathrm{e}^{i\alpha/2} \end{pmatrix}. \qquad (11)
\end{aligned}
$$

We can also obtain the form given in [1], which is called the Euler angle parametrization.

We conclude this note by making a comment on an application to quantum computation. We are interested in Quantum Computation. One of key points of quantum computation is to construct some unitary matrices (called quantum logic gates) in an efficient manner like Discrete Fourier Transform when $n$ is large enough. However, such a quick construction is not easy, see [3], [4], or see [5], [6], [7] for a qudit case. The parametrization of unitary matrices given by Jarlskog may be convenient for our real purpose.

We also make another comment that the parametrization method is similar to the idea in [8], namely the $(j-1)\times(j-1)$ block in (10) can be written as

$$
\begin{aligned}
\mathbf{1}_{j-1} - \left(1-\cos(\sqrt{\langle z_j|z_j\rangle})\right)|\tilde{z}_j\rangle\langle\tilde{z}_j| &= \mathbf{1}_{j-1} - |\tilde{z}_j\rangle\langle\tilde{z}_j| + \cos(\sqrt{\langle z_j|z_j\rangle})|\tilde{z}_j\rangle\langle\tilde{z}_j| \\
&= P_{j-1}(0) + \cos(\theta_{j-1})P_{j-1}(1)
\end{aligned}
$$

with the notations in [8]. Although $P_{j-1}(0)+\cos(\theta_{j-1})P_{j-1}(1)$ is of course not $P_{j-1}(0)+\mathrm{e}^{-i\theta_{j-1}}P_{j-1}(1)$, the method has a similar structure. Further study will be required.

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